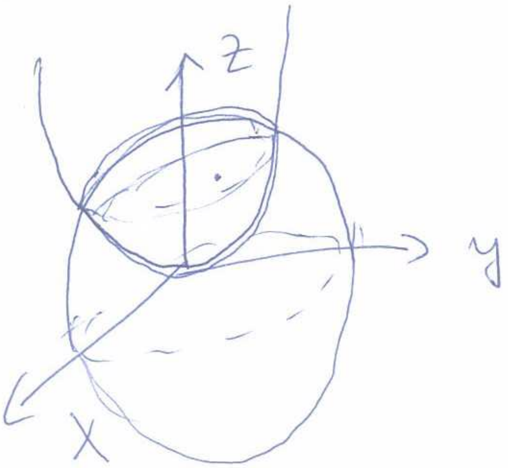
Primjer 4: Izračunati zapreminu tijela V ograničenog sferom $x^2+y^2+z^2=4$ i paraboloidom $3z=x^2+y^2$ unutar paraboloida.



(V) - tijelo ograničeno sferom i paraboloidom

V → zapremina tijela V.

$$\text{Iz} \quad \begin{cases} x^2+y^2+z^2=4 \\ 3z=x^2+y^2, \end{cases} \xrightarrow{(z>0)!} \text{dobijamo} \quad z^2+3z-4=0$$

$$\boxed{z_1=1}, \quad z_2=-4.$$

Dakle, projekcija tijela (V) na ravan Oxy je krug $K=\{(x,y) \mid x^2+y^2 \leq 3\}$ pa je:

$$V=\left\{(x,y,z) \mid x^2+y^2\leq 3, \ \frac{x^2+y^2}{3} \leq z \leq \sqrt{4-x^2-y^2}\right\}$$

Uvedimo cilindrične koordinate:

$$x=\rho\cos\varphi$$
$$y=\rho\sin\varphi \qquad (J=\rho).$$
$$z=z$$

Iz $\rho^2\cos^2\varphi+\rho^2\sin^2\varphi \leq 3$ dobijamo $\rho^2\leq 3$ tj. $0\leq\rho\leq\sqrt{3}$; dok je $0\leq\varphi\leq 2\pi$ i

$$\frac{\rho^2\cos^2\varphi+\rho^2\sin^2\varphi}{3} \leq z \leq \sqrt{4-\rho^2\sin^2\varphi-\rho^2\sin^2\varphi}$$

tj. $\dfrac{\rho^2}{3} \leq z \leq \sqrt{4-\rho^2}$.

Dakle, $(V'): \begin{cases} 0 \le \rho \le \sqrt{3} \\ 0 \le \varphi \le 2\pi \\ \frac{\rho^2}{3} \le z \le \sqrt{4-\rho^2} \end{cases}$

pa je $V = \iiint\limits_{(V)} dx\,dy\,dz = \iiint\limits_{(V')} \rho\, d\rho\, d\varphi\, dz =$

$$= \int_0^{\sqrt{3}} d\rho \int_0^{2\pi} d\varphi \int_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}} \rho\, dz = \int_0^{\sqrt{3}} d\rho \int_0^{2\pi} \rho\left(\sqrt{4-\rho^2} - \frac{\rho^2}{3}\right) d\varphi$$

$$= \int_0^{\sqrt{3}} \rho\left(\left(\sqrt{4-\rho^2}\right) - \frac{\rho^2}{3}\right) d\rho \int_0^{2\pi} d\varphi =$$

$$= 2\pi \int_0^{\sqrt{3}} \left(\rho\sqrt{4-\rho^2} - \frac{\rho^3}{3}\right) d\rho =$$

$$= 2\pi \cdot \frac{19}{12} = \frac{19\pi}{6}$$

$$\int_0^{\sqrt{3}} \left(\rho\sqrt{4-\rho^2} - \frac{\rho^3}{3}\right) d\rho = \left(-\frac{1}{3}(4-\rho^2)^{\frac{3}{2}} - \frac{\rho^4}{12}\right)\Big|_0^{\sqrt{3}} =$$

$$= \frac{19}{12}$$

$$\left[\int \rho\sqrt{4-\rho^2} = \left[\begin{array}{l} 4-\rho^2 = t \\ \rho\, d\rho = -\frac{1}{2} dt \end{array}\right] = -\frac{1}{2}\int \sqrt{t}\, dt = \right.$$

$$= -\frac{1}{2} \cdot \frac{2}{3} t^{\frac{3}{2}} + C$$

$$= -\frac{1}{3}(4-\rho^2)^{\frac{3}{2}} + C$$

Primjer 5: Nađi koordinate težišta homogenog tijela ograničenog ravninama:
$x=0,\ y=0,\ z=0,\ x=2,\ y=4,\ x+y+z=8.$

R) $(V)=\{(x,y,z)\mid 0\le x\le 2,\ 0\le y\le 4,\ 0\le z\le 8-x-y\}$

$$V=\iiint\limits_{(V)} dx\,dy\,dz=\int_0^2 dx\int_0^4 dy\int_0^{8-x-y} dz=$$
$$=\int_0^2 dx\int_0^4 \left(z\Big|_0^{(8-x-y)}\right)dy=$$
$$=\int_0^2 dx\int_0^4 (8-x-y)\,dy=$$
$$=\int_0^2\left(\left(8y-xy-\frac{y^2}{2}\right)\Big|_0^4\right)dx=$$
$$=\int_0^2\left(32-4x-\frac{16}{2}\right)dx=\int_0^2(24-4x)\,dx=$$
$$=\left(24x-\frac{4x^2}{2}\right)\Big|_0^2=48-\frac{4\cdot 4}{2}=40.$$

$$\iiint\limits_{(V)} x\,dx\,dy\,dz=\int_0^2 x\,dx\int_0^4 dy\int_0^{8-x-y} dz=$$
$$=\int_0^2 x\,dx\int_0^4(8-x-y)\,dy=\int_0^2 x\left(8y-xy-\frac{y^2}{2}\Big|_0^4\right)dx$$
$$=\int_0^2 x(24-4x)\,dx=\frac{112}{3}$$

$$\iiint\limits_{(V)} y\,dx\,dy\,dz=\int_0^2 dx\int_0^4 y\,dy\int_0^{8-x-y} dz=$$
$$=\int_0^2 dx\int_0^4 y\left(z\Big|_0^{8-x-y}\right)dy=$$

$$= \int_0^2 dx \int_0^4 (8-x-y)\,y\,dy =$$

$$= \int_0^2 dx \int_0^4 (8y - xy - y^2)\,dy = \int_0^2 \left(8\frac{y^2}{2} - x\frac{y^2}{2} - \frac{y^3}{3}\Big|_0^4\right) dx$$

$$= \int_0^2 \left(64 - 8x - \frac{64}{3}\right) dx = \int_0^2 \left(\frac{128}{3} - 8x\right) dx = \frac{208}{3}$$

slično,

$$\iiint_{(V)} z\,dx\,dy\,dz = \int_0^2 dx \int_0^4 dy \int_0^{8-x-y} z\,dz = \frac{320}{3}$$

pa je

$$x_T = \frac{\iiint_{(V)} x\,dx\,dy\,dz}{\iiint_{(V)} dx\,dy\,dz} = \frac{\frac{112}{3}}{40} = \frac{14}{15}$$

$$y_T = \frac{\iiint_{(V)} y\,dx\,dy\,dz}{\iiint_{(V)} dx\,dy\,dz} = \frac{26}{15}$$

$$z_T = \frac{\iiint_{(V)} z\,dx\,dy\,dz}{\iiint_{(V)} dx\,dy\,dz} = \frac{8}{3}$$

Dakle, $T\left(\frac{14}{15}, \frac{26}{15}, \frac{8}{3}\right)$ je težište tijela (V).